

## IB Linear Algebra – Example Sheet 1

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1. Suppose that the vectors  $\mathbf{e}_1, \dots, \mathbf{e}_n$  form a basis for a real vector space  $V$ . Which of the following are also bases?
  - (a)  $\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3, \dots, \mathbf{e}_{n-1} + \mathbf{e}_n, \mathbf{e}_n$ ;
  - (b)  $\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3, \dots, \mathbf{e}_{n-1} + \mathbf{e}_n, \mathbf{e}_n + \mathbf{e}_1$ ;
  - (c)  $\mathbf{e}_1 - \mathbf{e}_n, \mathbf{e}_2 + \mathbf{e}_{n-1}, \dots, \mathbf{e}_n + (-1)^n \mathbf{e}_1$ .
2. Let  $T, U$  and  $W$  be subspaces of  $V$ .
  - (i) Show that  $T \cup U$  is a subspace of  $V$  only if either  $T \leq U$  or  $U \leq T$ .
  - (ii) Give explicit counter-examples to the following statements:
    - (a)  $T + (U \cap W) = (T + U) \cap (T + W)$ ;
    - (b)  $(T + U) \cap W = (T \cap W) + (U \cap W)$ .
  - (iii) Show that each of the equalities in (ii) can be replaced by a valid inclusion of one side in the other.
3. For each of the following pairs of vector spaces  $(V, W)$  over  $\mathbb{R}$ , either give an isomorphism  $V \rightarrow W$  or show that no such isomorphism can exist. [Here  $P(\mathbb{R})$  denotes the space of polynomial functions  $\mathbb{R} \rightarrow \mathbb{R}$ , and  $C[a, b]$  denotes the space of continuous functions defined on the closed interval  $[a, b]$ .]
  - (a)  $V = \mathbb{R}^4$ ,  $W = \{\mathbf{x} \in \mathbb{R}^5 : x_1 + x_2 + x_3 + x_4 + x_5 = 0\}$ .
  - (b)  $V = \mathbb{R}^5$ ,  $W = \{p \in P : \deg p \leq 5\}$ .
  - (c)  $V = C[0, 1]$ ,  $W = C[-1, 1]$ .
  - (d)  $V = C[0, 1]$ ,  $W = \{f \in C[0, 1] : f(0) = 0, f$  continuously differentiable  $\}$ .
  - (e)  $V = \mathbb{R}^2$ ,  $W = \{\text{real solutions of } \ddot{x}(t) + x(t) = 0\}$ .
  - (f)  $V = \mathbb{R}^4$ ,  $W = C[0, 1]$ .
  - (g) (Harder:)  $V = P(\mathbb{R})$ ,  $W = \mathbb{R}^{\mathbb{N}}$ .
4. (i) If  $\alpha$  and  $\beta$  are linear maps from  $U$  to  $V$  show that  $\alpha + \beta$  is linear. Give explicit counter-examples to the following statements:
  - (a)  $\text{im}(\alpha + \beta) = \text{im}(\alpha) + \text{im}(\beta)$ ;
  - (b)  $\text{Ker}(\alpha + \beta) = \text{Ker}(\alpha) \cap \text{Ker}(\beta)$ .

Show that in general each of these equalities can be replaced by a valid inclusion of one side in the other.

(ii) Let  $\alpha$  be a linear map from  $V$  to  $V$ . Show that if  $\alpha^2 = \alpha$  then  $V = \text{Ker}(\alpha) \oplus \text{im}(\alpha)$ . Does your proof still work if  $V$  is infinite dimensional? Is the result still true?
5. Let
 
$$U = \{\mathbf{x} \in \mathbb{R}^5 : x_1 + x_3 + x_4 = 0, 2x_1 + 2x_2 + x_5 = 0\}, \quad W = \{\mathbf{x} \in \mathbb{R}^5 : x_1 + x_5 = 0, x_2 = x_3 = x_4\}.$$

Find bases for  $U$  and  $W$  containing a basis for  $U \cap W$  as a subset. Give a basis for  $U + W$  and show that

$$U + W = \{\mathbf{x} \in \mathbb{R}^5 : x_1 + 2x_2 + x_5 = x_3 + x_4\}.$$

6. (i) Let  $\alpha : V \rightarrow V$  be an endomorphism of a finite dimensional vector space  $V$ . Show that

$$V \geq \text{im}(\alpha) \geq \text{im}(\alpha^2) \geq \dots \quad \text{and} \quad \{0\} \leq \text{Ker}(\alpha) \leq \text{Ker}(\alpha^2) \leq \dots$$

If  $r_k = \text{rk}(\alpha^k)$ , deduce that  $r_k \geq r_{k+1}$  and that  $r_k - r_{k+1} \geq r_{k+1} - r_{k+2}$ . Conclude that if, for some  $k \geq 0$ , we have  $r_k = r_{k+1}$ , then  $r_k = r_{k+\ell}$  for all  $\ell \geq 0$ .

- (ii) Suppose that  $\dim(V) = 5$ ,  $\alpha^3 = 0$ , but  $\alpha^2 \neq 0$ . What possibilities are there for  $\text{rk}(\alpha)$  and  $\text{rk}(\alpha^2)$ ?

7. Let  $\alpha: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be the linear map given by  $\alpha: \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ . Find the matrix representing  $\alpha$  relative to the basis  $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$  for both the domain and the range.

Write down bases for the domain and range with respect to which the matrix of  $\alpha$  is the identity.

8. Let  $U_1, \dots, U_k$  be subspaces of a vector space  $V$  and let  $B_i$  be a basis for  $U_i$ . Show that the following statements are equivalent:

- (i)  $U = \sum_i U_i$  is a direct sum, i.e. every element of  $U$  can be written uniquely as  $\sum_i u_i$  with  $u_i \in U_i$ .
- (ii)  $U_j \cap \sum_{i \neq j} U_i = \{0\}$  for all  $j$ .
- (iii) The  $B_i$  are pairwise disjoint and their union is a basis for  $\sum_i U_i$ .

Give an example where  $U_i \cap U_j = \{0\}$  for all  $i \neq j$ , yet  $U_1 + \dots + U_k$  is not a direct sum.

9. Show that any two subspaces of the same dimension in a finite dimensional vector space have a common complementary subspace.

10. Let  $Y$  and  $Z$  be subspaces of the finite dimensional vector spaces  $V$  and  $W$ , respectively. Show that  $R = \{\alpha \in \mathcal{L}(V, W) : \alpha(Y) \leq Z\}$  is a subspace of the space  $\mathcal{L}(V, W)$  of all linear maps from  $V$  to  $W$ . What is the dimension of  $R$ ?

11. Let  $Y$  and  $Z$  be subspaces of the finite dimensional vector spaces  $V$  and  $W$  respectively. Suppose that  $\alpha: V \rightarrow W$  is a linear map such that  $\alpha(Y) \subset Z$ . Show that  $\alpha$  induces linear maps  $\alpha|_Y: Y \rightarrow Z$  via  $\alpha|_Y(y) = \alpha(y)$  and  $\bar{\alpha}: V/Y \rightarrow W/Z$  via  $\bar{\alpha}(v+Y) = \alpha(v)+Z$ . Consider a basis  $(v_1, \dots, v_n)$  for  $V$  containing a basis  $(v_1, \dots, v_k)$  for  $Y$  and a basis  $(w_1, \dots, w_m)$  for  $W$  containing a basis  $(w_1, \dots, w_l)$  for  $Z$ . Show that the matrix representing  $\alpha$  with respect to  $(v_1, \dots, v_n)$  and  $(w_1, \dots, w_m)$  is a block matrix of the form  $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ . Explain how to determine the matrices representing  $\alpha|_Y$  with respect to the bases  $(v_1, \dots, v_k)$  and  $(w_1, \dots, w_l)$  and representing  $\bar{\alpha}$  with respect to the bases  $(v_{k+1} + Y, \dots, v_n + Y)$  and  $(w_{l+1} + Z, \dots, w_m + Z)$  from this block matrix.

12. Let  $T, U, V, W$  be vector spaces over  $\mathbb{F}$  and let  $\alpha: T \rightarrow U$ ,  $\beta: V \rightarrow W$  be fixed linear maps. Show that the mapping  $\Phi: \mathcal{L}(U, V) \rightarrow \mathcal{L}(T, W)$  which sends  $\theta$  to  $\beta \circ \theta \circ \alpha$  is linear. If the spaces are finite-dimensional and  $\alpha$  and  $\beta$  have rank  $r$  and  $s$  respectively, find the rank of  $\Phi$ .